This week

1. Appendix A.7: complex numbers
2. Application: impedance

*The Riemann zeta-function*
A complex number is a vector in $\mathbb{R}^2$.

In stead of $\mathbb{R}^2$ we write $\mathbb{C}$.

Rather than $x$- and $y$-axis, we call them the real axis and imaginary axis.

The complex number $i$ is defined as $(0, 1)$.

Addition is defined termwise: if $z = (x, y)$ and $w = (u, v)$, then

$$z + w = (x + y, y + v)$$

Scalar multiplication is defined termwise: if $z = (x, y)$ and $\alpha \in \mathbb{R}$, then

$$\alpha z = (\alpha x, \alpha y)$$
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The product of \( z \) and \( w \) is defined as

\[
z \cdot w = (x \cdot u - y \cdot v, x \cdot v + y \cdot u)
\]

Examples:

- \((1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)\).
- \((2, 0)(3, -4) = (2 \cdot 3 - 0(-4), 2(-4) + 0 \cdot 3) = (2 \cdot 3, 2(-4)) = 2(3, -4)\).
- \(i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)\).

The real axis

Convention

Every real number \( x \) is identified with the complex number \((x, 0)\).

Examples: \( 0 = (0, 0), 1 = (1, 0), -1 = (-1, 0)\).

The complex numbers on the real axis behave just like the real numbers in \( \mathbb{R} \):

- \( x + y \rightarrow (x, 0) + (y, 0) = (x + y, 0 + 0) = (x + y, 0) \).
- \( x - y \rightarrow (x, 0) - (y, 0) = (x - y, 0 - 0) = (x - y, 0) \).
- \( xy \rightarrow (x, 0)(y, 0) = (xy - 0 \cdot 0, x \cdot 0 + 0 \cdot y) = (xy, 0) \).
Real numbers are complex numbers

- By identifying \( x \in \mathbb{R} \) with the complex number \((x, 0)\), we regard the points on the real axis as the real number line.

\[ i^2 = -1 \]

- The complex numbers are an expansion of the real numbers:

\[ \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \]

Algebraic laws for sum, difference and product

- Let \( z, w \) and \( u \) be complex numbers. Define \( z - w \) and \( -z \) in the usual way, then

1. \( z + w = w + z \)
2. \( z + w + u = z + (w + u) = (z + w) + u \)
3. \( z + 0 = z \)
4. \( -z = (-1)z \)
5. \( z - w = z + (-w) \)
6. \( z - z = 0 \)
7. \( zw = wz \)
8. \( z \cdot 1 = z \)
9. \( z \cdot 0 = 0 \)
10. \( zwu = z(wu) = (zw)u \)
11. \( z(w + u) = zw + zu \)
12. \( z(w - u) = zw - zu \)
The canonical form

**Theorem**

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
z = x + i \, y.
\]

**Proof:**

\[
x + i \, y = (x, 0) + (0, 1)(y, 0)
= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)
= (x, 0) + (0, y) = (x, y) = z.
\]

**Definition**

The form \( x + i \, y \) is called the **canonical form** of \( z \).

Henceforth we will always write complex numbers in canonical form.

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**Canonical form for sum and multiplication**

- Let \( z = x + i \, y \) and \( w = u + i \, v \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w = (x + i \, y) + (u + i \, v)
= x + u + i \, y + i \, v
= (x + u) + i(y + v).
\]

- For the product of \( z \) and \( w \) we have

\[
z \, w = (x + i \, y)(u + i \, v)
= x \, u + (i \, y)(i \, v) + x(i \, v) + (i \, y)u
= x \, u + i^2y \, v + i \, x \, v + i \, y \, u
= (x \, u - y \, v) + i(x \, v + y \, u).
\]
### Definition

Let $z = x + iy$ be a complex number with $x$ and $y$ real. Then $x$ is the **real part** of $z$ and $y$ is the **imaginary part** of $z$. We denote

$$x = \Re z \quad \text{and} \quad y = \Im z.$$
The absolute value

**Definition**

Let $z = x + iy$ be a complex number with $x$ and $y$ real. Then the absolute value of $z$ is the distance of $z$ to 0:

$$|z| = \sqrt{x^2 + y^2}.$$

- The definition is based on the Pythagorean theorem.
- The absolute value is sometimes called **modulus** or **norm**.

Properties of conjugation and absolute value

- Let $z$ and $w$ be complex numbers, then
  1. $\overline{z + w} = \overline{z} + \overline{w}$
  2. $\overline{z - w} = \overline{z} - \overline{w}$
  3. $\overline{zw} = \overline{z} \overline{w}$
  4. $|z|^2 = z \overline{z}$
  5. $|zw| = |z| |w|$
  6. $|z + w| \leq |z| + |w|$

- Property 6 is called the **triangular inequality**.
The real and imaginary part

**Theorem**

*For every complex number* \( z \) *the following holds:*

1. \( \text{Re} \, z = \frac{z + \bar{z}}{2} \)
2. \( \text{Im} \, z = \frac{z - \bar{z}}{2} \, i \)

■ Write \( z = x + iy \), then

\[
\begin{align*}
z + \bar{z} &= (x + iy) + (x - iy) = 2x = 2 \text{Re} \, z, \\
z - \bar{z} &= (x + iy) - (x - iy) = 2iy,
\end{align*}
\]

\[
-\frac{1}{2} i (z - \bar{z}) = y = \text{Im} \, z.
\]

**Exercises**

Assignment: **IMM2 - Tutorial 6.1**
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

$$\overline{w} z w = \overline{z} \Rightarrow |z|^2 w = \overline{z} \Rightarrow \frac{1}{z} = w = \frac{1}{|z|^2} \overline{z}$$

- The number $w$ is called the \textbf{reciprocal of} $z$ and is denoted as $\frac{1}{z}$.

- The reciprocal of $z$ is sometimes denoted as $z^{-1}$.

- If $z = x + iy$ with $x$ and $y$ real, then

$$\frac{1}{z} = \frac{1}{|z|^2} \overline{z} = \frac{1}{x^2 + y^2} (x - iy) = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i.$$ 

## Division

### Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the \textbf{quotient of} $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \cdot \frac{1}{z}.$$  

- Equivalently we can write $\frac{u}{z} = \frac{1}{|z|^2} u \overline{z}$.

- Practical approach: multiply numerator and denominator with $\overline{z}$:

$$u = \frac{u \overline{z}}{z \overline{z}}$$

and elaborate $u \overline{z}$.

- Example:

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{5} = 1 - i.$$
Algebraic laws for quotient

Let \( u \neq 0, \ v, \ z \neq 0 \) and \( w \) be complex numbers.

1. \( \frac{w}{1} = w \)

2. \( \frac{w}{z} \frac{v}{u} = \frac{w v}{z u} \)

3. \( \frac{1}{w/z} = \frac{z}{w} \) (for \( w \neq 0 \))

4. \( \frac{w}{z} = \frac{\bar{w}}{\bar{z}} \)

5. \( \frac{|w|}{|z|} = \frac{w}{z} \)

For all \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z} \) the following holds:

1. \( z^m z^n = z^{m+n} \)

2. \( (z^m)^n = z^{mn} \)

3. \( \frac{1}{z^m} = z^{-m} \)

4. \( z^n w^n = (zw)^n \)

5. \( \left( \frac{w}{z} \right)^n = \frac{w^n}{z^n} \)

Exercises

Assignment: IMM2 - Tutorial 6.2
The argument

**Definition**

The argument of a complex number $z \neq 0$ is the angle that the line through 0 and $z$ makes with the positive real axis.

The argument of $z$ is denoted as $\arg(z)$.

- The argument of 0 is not defined.
- The argument is expressed in radians.
- The argument is measured from the positive real axis.
- If the direction is counter-clockwise, the argument is positive.
- If the direction is clockwise, the argument is negative.
- The argument is determined up to a multiple of $2\pi$.

The Euler function

**Definition**

The Euler function is the function that assigns to every real number $\varphi$ the complex number

$$e^{i\varphi} = \cos \varphi + i \sin \varphi.$$

- The number $e^{i\varphi}$ lies on the unit circle: $|e^{i\varphi}| = 1$.
- The real part of $e^{i\varphi}$ is $\cos \varphi$, the imaginary part of $e^{i\varphi}$ is $\sin \varphi$.
- The complex number $e^{i\varphi}$ is the number on the unit circle with argument $\varphi$. 
Theorem

For every real number $\varphi$ and $\psi$ we have

$$e^{i(\varphi+\psi)} = e^{i\varphi} e^{i\psi}$$

- Use trigonometry formulas to derive

$$e^{i(\varphi+\psi)} = \cos(\varphi + \psi) + i \sin(\varphi + \psi)$$
$$= \cos \varphi \cos \psi - \sin \varphi \sin \psi + i (\sin \varphi \cos \psi + \cos \varphi \sin \psi).$$

- Expand the right-hand side:

$$e^{i\varphi} e^{i\psi} = (\cos \varphi + i \sin \varphi) (\cos \psi + i \sin \psi)$$
$$= \cos \varphi \cos \psi + i^2 \sin \varphi \sin \psi + i \sin \varphi \cos \psi + i \cos \varphi \sin \psi$$
$$= \cos \varphi \cos \psi - \sin \varphi \sin \psi + i (\sin \varphi \cos \psi + \cos \varphi \sin \psi)$$
$$= e^{i(\varphi+\psi)}.$$

The Simpsons (from episode ‘Treehouse of Horror VI’)

The Euler function Cheat Sheet

1. $e^{i0} = 1$
2. $|e^{i\varphi}| = 1$
3. $e^{i(\varphi+\psi)} = e^{i\varphi} e^{i\psi}$
4. $(e^{i\varphi})^n = e^{in\varphi}$ for all $n \in \mathbb{Z}$.
5. $e^{i\varphi} = e^{-i\varphi} = \frac{1}{e^{i\varphi}}$
### Theorem

Every complex number \( z \neq 0 \) can be written as the product of a positive real number and an Euler function value. In particular, if \( r = |z| \) and \( \varphi = \arg z \), then

\[
z = r e^{i\varphi}
\]

- Write \( z = x + i y \) with \( x \) and \( y \) real, then
  \[
  \cos \varphi = \frac{x}{r} \quad \text{and} \quad \sin \varphi = \frac{y}{r}.
  \]
- \( z = x + i y \)
  \[
  = r \cos \varphi + i(r \sin \varphi)
  = r(\cos \varphi + i \sin \varphi) = re^{i\varphi}.
  \]

### Theorem

Let \( z \) and \( w \) be two complex numbers written in polar coordinates:

\[
z = r e^{i\varphi} \quad \text{and} \quad w = s e^{i\psi},
\]

then

\[
z w = rs e^{i(\varphi+\psi)} \quad \text{and (if } w \neq 0) \quad \frac{z}{w} = \frac{r}{s} e^{i(\varphi-\psi)}.
\]

- In other words:
  - the absolute value of \( z w \) is the **product** of \( |z| \) and \( |w| \),
  - the argument of \( z w \) is the **sum** of \( \arg z \) and \( \arg w \), and:
  - the absolute value of \( z/w \) is the **quotient** of \( |z| \) and \( |w| \),
  - the argument of \( z/w \) is the **difference** of \( \arg z \) and \( \arg w \).
**Corollary**

Let \( w = r e^{i\varphi} \). Then multiplication of an arbitrary complex number \( z \) with \( w \) can be constructed geometrically by scaling \( z \) with scale factor \( r \), and by rotating \( z \) over an angle \( \varphi \) about 0.

- **Example**: let \( w = \sqrt{3} + i = 2e^{i\pi/6} \), then \( zw \) is obtained by scaling \( z \) with factor 2, and by rotating \( z \) over an angle of 30°.

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**Exercises**

**Assignment**: IMM2 - **Tutorial 6.3**

- **Rectangular form** is the same as canonical form.
- **Trigonometric form** is like polar form but with sine and cosine, and with a non-negative angle smaller than 360°, for example:

  \[
  z = 7\left(\cos(225°) + i\sin(225°)\right).
  \]

- MyLabsPlus uses \( \text{cis} \) (“cosine plus i sine”) to indicate Eulers function:

  \[
  \text{cis}(\varphi) = e^{i\varphi}.
  \]
In this part of the lecture we write $j$ in stead of $i$. 

### Passive components

<table>
<thead>
<tr>
<th>Component</th>
<th>Relation ( v(t) ) vs. ( i(t) )</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>[ v(t) = R i(t) ]</td>
<td>Dissipates energy</td>
</tr>
<tr>
<td>Capacitor</td>
<td>[ v(t) = \frac{1}{C} \int_{0}^{t} i(\tau) , d\tau ] [ i(t) = C v'(t) ]</td>
<td>Stores energy in an electric field</td>
</tr>
<tr>
<td>Inductor</td>
<td>[ v(t) = L i'(t) ]</td>
<td>Stores energy in a magnetic field</td>
</tr>
</tbody>
</table>
Passive components

4.3

If you know $i(t)$, then $v(t)$ can be uniquely determined.

The component can therefore be regarded to be a **system**:

$$i(t) \rightarrow S \rightarrow v(t)$$

or abbreviated: $i(t) \mapsto v(t)$.

**Example**: for an inductor with inductance $L$ we have

$$i(t) \mapsto Li'(t).$$

Complex signals

4.4

**Definition**

Let $S$ be a system. Let $x(t)$ and $y(t)$ be signals for which $S$ has the following responses:

$$x(t) \mapsto u(t) \quad \text{and} \quad y(t) \mapsto v(t).$$

Then the response of $S$ to the input $x(t) + jy(t)$ is defined as

$$u(t) + jv(t).$$

**Example**: for an inductor with inductance $L$ we have

$$\cos(\omega t) \mapsto -\omega L \sin(\omega t) \quad \text{and} \quad \sin(\omega t) \mapsto \omega L \cos(\omega t),$$

hence

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \mapsto -\omega L \sin(\omega t) + j\omega L \cos(\omega t)$$

$$= j\omega L \left( \cos(\omega t) + j \sin(\omega t) \right)$$

$$= j\omega L e^{j\omega t}.$$
Theorem

Passive components are linear and time invariant.

- Linearity means that if $x_1(t) \mapsto y_1(t)$ and $x_2(t) \mapsto y_2(t)$, then 
  $$\alpha x_1(t) + \beta x_2(t) \mapsto \alpha y_1(t) + \beta y_2(t).$$
  for all $\alpha$ and $\beta$.

- Time invariance means that if $x(t) \mapsto y(t)$, then 
  $$x(t - t_0) \mapsto y(t - t_0)$$
  for all $t_0$.

- Linear and time invariant systems are called LTI systems.
- Passive components can be regarded as systems: the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component.
- Passive components are LTI systems.

The transfer function

For all LTI systems there exists a function $Z(\omega)$ such that 

$$e^{j\omega t} \mapsto Z(\omega) e^{j\omega t}$$

- The function $Z(\omega)$ is called the transfer function.
- The transfer function does not depend on time, but can depend on the frequency $\omega$.
- For passive components, where the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component, the function $Z(\omega)$ is called the impedance of the component, usually denoted as $Z$.
- Example: for an inductor with inductance $L$ we have 
  $$e^{j\omega t} \mapsto j\omega L e^{j\omega t},$$
  so the impedance is $Z = j\omega L$. 
Impedance

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Example

Let $v(t) = 5\cos(2\pi f t)$, where the frequency is equal to $f = 10$ kHz. The inductance $L$ is 50 mH. Describe the current $i(t)$ through the inductor as a function of $t$. What is the amplitude of $i(t)$?

- The impedance of $L$ is $Z = j\omega L$, where $\omega = 2\pi f$.
- Define $\dot{v}(t) = 5e^{j\omega t}$, then
  $$\dot{i}(t) = \frac{\ddot{v}(t)}{Z} = \frac{\ddot{v}(t)}{j\omega L} = -\frac{5j e^{j\omega t}}{\omega L} = -\frac{5j}{2\pi f L} \left( \cos(\omega t) + j\sin(\omega t) \right)$$
  $$= \frac{5}{2\pi f L} \left( \sin(\omega t) - j\cos(\omega t) \right).$$
- Hence $i(t) = \text{Re}(\dot{i}(t)) = \frac{5}{2\pi f L} \sin(2\pi f t) \approx 0.001591 \sin(2\pi f t)$.
- The amplitude of $i(t)$ is 1.591 mA.
Composition in series

- The following relations hold:
  \[ v_1(t) = Z_1 i(t) \quad \text{and} \quad v_2(t) = Z_2 i(t). \]
- The voltage over clamps \(AB\) is
  \[ v(t) = v_1(t) + v_2(t) = Z_1 i(t) + Z_2 i(t) = (Z_1 + Z_2)i(t). \]
- The replacement impedance for the series composition is \(v(t)/i(t)\), hence
  \[ Z_{\text{ser}} = Z_1 + Z_2. \]

Composition in parallel

- The total current through the circuit is
  \[ i(t) = i_1(t) + i_2(t) = \frac{v(t)}{Z_1} + \frac{v(t)}{Z_2} = \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right) v(t). \]
- The reciprocal of the replacement impedance is \(i(t)/v(t)\), hence
  \[ \frac{1}{Z_{\text{par}}} = \frac{1}{Z_1} + \frac{1}{Z_2}. \]
Example 4.11

- The for the impedance of an inductor $L$ parallel to a capacitor $C$ we have

$$\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{1/j\omega C} = \frac{1}{j\omega L} + j\omega C = \frac{1 - \omega^2 LC}{j\omega L}.$$ 

- The impedance of the circuit is

$$Z = \frac{j\omega L}{1 - \omega^2 LC}.$$

The resonance frequency 4.12

- The impedance becomes very large if $\omega^2 \approx \frac{1}{LC}$.  
- The frequency $\omega_{res} = \frac{1}{\sqrt{LC}}$ is called the resonance frequency.
Answer the following questions for the circuits (1), (2) and (3).

(a) What is the replacement impedance $Z$? Write $Z$ in canonical form.

(b) What happens with $|Z|$ for high frequencies ($\omega \to \infty$)?

(c) What happens with $|Z|$ for low frequencies ($\omega \to 0$)?